BULGARIAN NATIONAL MATHEMATICAL OLYMPIAD FOR UNIVERSITY STUDENTS

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GROUP A

Problem 1. Let A be a real square matrix of n-th order, for which $\det A \neq 0$ and $A^3 + A = \mathbf{O}$.

a) Solve the equation $XA = A^{-1}$.

b) Prove that $\left(\frac{1+i}{1-i}\right)^{2021}$ is a characteristic root (eigenvalue) of matrix A. Give an example of such a matrix of 2-nd order.

c) Is it possible for the order n of A to be 2021? Prove if it is not possible and if it is possible, give an example.

($\mathbf{O}$ is the zero matrix).

Solution: From the condition we have that $A(A^2 + E) = \mathbf{O}$ (E is the identity matrix of n-th order) and the matrix A is reversible. Therefore $A^2 = -E$ and $A^{-1} = -A$.

a) The equation from the condition is equivalent to $X\,A = -A$.

A matrix is multiplied either by a matrix of appropriate dimension or by a number. So $X = -E$ or $X = -1$.

If λ is a characteristic root of A, then $\lambda^3 + \lambda = 0$ and $\lambda \neq 0$ ($\det A \neq 0$). Consequently $\lambda = \pm i$. Because it is a real matrix, its characteristic roots are two by two conjugated. In particular, each of the numbers i and $-i$ is a characteristic root.

b) $\left(\frac{1+i}{1-i}\right)^{2021} = \left(\frac{(1+i)^2}{2}\right)^{2021} = i^{2021} = i$ is a characteristic root. One matrix satisfying the condition is $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

c) Because the characteristic roots of the matrix A are conjugated in pairs, its order n is an even number.

Problem 2. Let A and B be square matrices with real elements of order n, for which $A^2 = A$, $B^2 = B$ and $A\,B = B\,A$. Prove that $AB = \mathbf{O}$ if and only if $\mathbf{r}(A+B) = \mathbf{r}(A) + \mathbf{r}(B)$.

($\mathbf{O}$ is the zero matrix and $\mathbf{r}(A)$ is the rank of the matrix A)

Solution: Let φ and ψ are the linear operators in the space $\mathbb{V} = \mathbb{R}^n$ with matrices A and B in a standard basis of $\mathbb{R}^n$ (for a vector column $v \in \mathbb{R}^n$ $\varphi(v) = Av$ and $\psi(v) = Bv$). So $\varphi^2 = \varphi$,

$\psi^2=\psi$ and $\varphi\psi=\psi\varphi$. Let $\mathbb{U}_0=\mathrm{Ker}\,\varphi$ and $\mathbb{U}_1=\mathrm{Ker}(\varphi-\mathbf{id}_{\mathbb{V}})=\mathrm{Im}\,\varphi$. Therefore $\mathbb{U}_0\oplus\mathbb{U}_1=\mathbb{V}$ (for $v\in\mathbb{V}$ $v=(v-\varphi(v))+\varphi(v)$ and $v-\varphi(v)\in\mathbb{U}_0$, $\varphi(v)\in\mathbb{U}_1$ and $\mathbb{U}_0\cap\mathbb{U}_1=\{\mathbf{0}\}$) and $\mathbb{U}_0$ and $\mathbb{U}_1$ are φ- invariant. Thus for φ a basis exists, in witch its matrix is diagonal with 0 and 1, i.e. A is similar to a diagonal matrix of 0 and 1. Subspaces $\mathbb{U}_0$ and $\mathbb{U}_1$ are also ψ-invariant.

- If $v\in\mathbb{U}_0$ then $\varphi(v)=\mathbf{o}$. Then $\varphi(\psi(v))=\psi(\varphi(v))=\psi(\mathbf{o})=\mathbf{o}$ and $\psi(v)\in\mathbb{U}_0$.
- If $v\in\mathbb{U}_1$ then $\varphi(v)=v$. Then $\varphi(\psi(v))=\psi(\varphi(v))=\psi(v)$ and $\psi(v)\in\mathbb{U}_1$.

Let ψ_0 and ψ_1 are the restrictions of ψ over $\mathbb{U}_0$ and $\mathbb{U}_1$ respectively. Then they will be linear operators in those subspaces for which $\psi_0^2=\psi_0$ and $\psi_1^2=\psi_1$ and, therefore, are diagonalized. This means that there is a basis in which matrices φ and ψ are diagonal with 0 and 1 on the diagonal. Let these be the matrices P and Q. Then there is a reversible matrix T such that $A=TPT^{-1}$ and $B=TQT^{-1}$. Hence

$$AB=\mathbf{O}\Leftrightarrow PQ=\mathbf{O},\ \mathbf{r}(A)=\mathbf{r}(P),\ \mathbf{r}(B)=\mathbf{r}(Q)\ \text{и}\ \mathbf{r}(A+B)=\mathbf{r}(P+Q),$$

i.e. without any restrictions we can assume that A and B are diagonal with 0 and 1 on the diagonal.

Obviously $\mathbf{r}(P+Q)=\mathbf{r}(P)+\mathbf{r}(Q)$ if and only if $p_{ii}q_{ii}=0$ for each $i=1,\ 2,\ n$, which is equivalent to $PQ=\mathbf{O}$ (the ranks of the matrices P and Q are the number of their 1 on the diagonal).

Problem 3. The sequence $a_0=0$ and $a_n=2n(a_{n-1}+1)$ is given for $n\geq 1$. Find:

$$\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{a_n}\right).$$

Solution: Lets consider the partial sum

$$\sum_{n=1}^{N}\ln\left(1+\frac{1}{a_n}\right)=\ln\prod_{n=1}^{N}\left(1+\frac{1}{a_n}\right)=\ln\frac{a_{N+1}}{2^{N+1}(N+1)!}.$$

From the recurrent expression we obtain

$$\frac{a_{N+1}}{2^{N+1}(N+1)!}=\frac{a_{N+1}}{2(N+1)}\frac{1}{2^N N!}=\frac{a_N+1}{2^N N!}=\frac{a_N}{2^N N!}+\frac{1}{2^N N!}.$$

Inductive

$$\frac{a_{N+1}}{2^{N+1}(N+1)!}=\sum_{k=0}^{N}\frac{1}{2^k k!}.$$

Therefore

$$\lim_{N\to\infty}\sum_{n=1}^{N}\ln\left(1+\frac{1}{a_n}\right)=\ln\sum_{k=0}^{\infty}\frac{1}{2^k k!}=\ln\sqrt{e}=\frac{1}{2}.$$

Problem 4. Let $a_n = \sum_{k=2n}^{\infty} \frac{(-1)^k}{k}$.

a) Prove: $\int_n^{\infty} \frac{dx}{2x(2x+1)} < a_n < \int_{n-1}^{\infty} \frac{dx}{2x(2x+1)}$.

b) Find: $\lim_{n\to\infty} na_n$.

c) If $\lim_{n\to\infty} na_n = \alpha$, find: $\lim_{n\to\infty} n(na_n - \alpha)$.

Solution:

a) Following from

$$a_n = \sum_{k=n}^{\infty}\left(\frac{1}{2k} - \frac{1}{2k+1}\right) = \sum_{k=n}^{\infty} \frac{1}{2k(2k+1)}$$

and inequalities

$$\int_k^{k+1} \frac{dx}{2x(2x+1)} < \frac{1}{2k(2k+1)} < \int_{k-1}^{k} \frac{dx}{2x(2x+1)}.$$

b) From a) we have

$$\int_n^{\infty} \frac{dx}{(2x+1)^2} < \int_n^{\infty} \frac{dx}{2x(2x+1)} < a_n < \int_{n-1}^{\infty} \frac{dx}{2x(2x+1)} < \int_{n-1}^{\infty} \frac{dx}{(2x)^2},$$

i.e.

$$\frac{n}{2(2n+1)} < na_n < \frac{n}{4(n-1)}.$$

$$\lim_{n\to\infty} \frac{n}{2(2n+1)} = \lim_{n\to\infty} \frac{n}{4(n-1)} = \frac{1}{4} \;\Rightarrow\; \lim_{n\to\infty} na_n = \frac{1}{4}.$$

c) We will estimate $\left| \sum_{k=n}^{\infty} \frac{1}{2k(2k+1)} - \int_{n-\frac{1}{2}}^{\infty} \frac{dx}{2x(2x+1)} \right|$.

$$\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{dx}{2x(2x+1)} = \frac{1}{2}\ln \frac{2x}{2x+1}\Bigg|_{k-\frac{1}{2}}^{k+\frac{1}{2}} = \frac{1}{2}\ln\frac{2k^2+k}{2k^2+k-1} =$$

$$= \frac{1}{2}\ln\left(1+\frac{1}{2k^2+k-1}\right) = \frac{1}{2k(2k+1)} + O\left(\frac{1}{k^4}\right).$$

Then

$$\left| \sum_{k=n}^{\infty} \frac{1}{2k(2k+1)} - \int_{n-\frac{1}{2}}^{\infty} \frac{dx}{2x(2x+1)} \right| = O\left(\sum_{k=n}^{\infty} \frac{1}{k^4}\right) = O\left(\frac{1}{n^3}\right).$$

Consequently

$$a_n = \int_{n-\frac{1}{2}}^{\infty} \frac{dx}{2x(2x+1)} + \mathrm{O}\left(\frac{1}{n^3}\right) = \frac{1}{2}\ln\frac{2n}{2n-1} + \mathrm{O}\left(\frac{1}{n^3}\right) =$$

$$= \frac{1}{2}\ln\left(1+\frac{1}{2n-1}\right) + \mathrm{O}\left(\frac{1}{n^3}\right) = \frac{1}{2(2n-1)} - \frac{1}{4(2n-1)^2} + \mathrm{O}\left(\frac{1}{n^3}\right),$$

whence

$$\lim_{n\to\infty} n\left(na_n - \frac{1}{4}\right) = \lim_{n\to\infty}\left(\frac{n^2-n}{4(2n-1)^2}\right) = \frac{1}{16}.$$