BULGARIAN NATIONAL MATHEMATICAL OLYMPIAD FOR UNIVERSITY STUDENTS

Varna, 14-16th May, 2021

GROUP C

Problem 1. Let $MNPQ$ be a square. The points $A(2,1)$ and $B(3,5)$ lie respectively on lines MN and PQ and the points $C(0,1)$ and $D(-3,-1)$ lie respectively on lines MQ and NP. Find the equations of lines MN, PQ, MQ, and NP and the area of $MNPQ$.

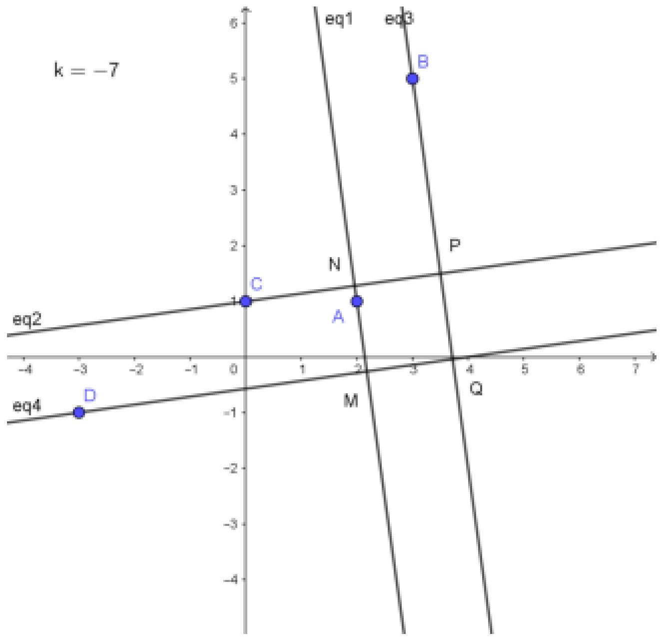
Solution: $A\in MN$, $B\in PQ$, $C\in NP$, $D\in MQ$. Then $MN: y-1=k(x-2)$, i.e. $MN: kx-y+1-2k=0$. The line NP goes through point C and it is perpendicular to the line MN, hence its slope is $-\frac{1}{k}$. For the equation of NP we obtain $NP: y-1=-\frac{1}{k}x$, i.e. $x+ky-k=0$.

To determine k we will use the property of the square: $d(B,MN)=d(D,NP)$.

$d(B,MN)=\frac{|3k-5+1-2k|}{\sqrt{k^2+1}}=\frac{|k-4|}{\sqrt{k^2+1}}$, and $d(D,NP)=\frac{|-3-k-k|}{\sqrt{1+k^2}}=\frac{|3+2k|}{\sqrt{1+k^2}}$. Therefore $\frac{|k-4|}{\sqrt{k^2+1}}=\frac{|3+2k|}{\sqrt{1+k^2}}$, $k-4=\pm(3+2k)$, $k=-7$ or $k=\frac{1}{3}$.

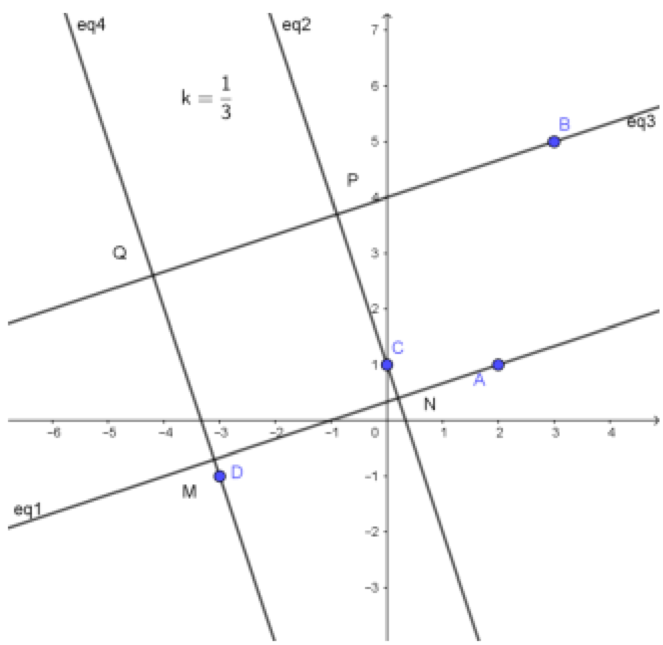
I) $k=-7$:

$MN: 7x+y-15=0$, $NP: x-7y+7=0$, $PQ: 7x+y-26=0$, $QM: x-7y-4=0$. Then $MN\cap NP=N\left(\frac{49}{25},\frac{32}{25}\right)$, $PQ\cap NP=P\left(\frac{7}{2},\frac{3}{2}\right)$ and $S=NP^2=\left(\frac{11\sqrt{2}}{10}\right)^2=\frac{121}{50}$.



II) $k=\frac{1}{3}$:

$MN: x-3y+1=0$, $NP: 3x+y-1=0$, $PQ: x-3y+12=0$, $QM: 3x+y+10=0$. Then $MN\cap NP=N\left(\frac{1}{5},\frac{2}{5}\right)$, $PQ\cap NP=P\left(-\frac{9}{10},\frac{37}{10}\right)$ and $S=NP^2=\left(\frac{11\sqrt{10}}{10}\right)^2=\frac{121}{10}$.



Problem 2. The functions $f(x)$ and $g(x)$ are given:

$$f(x)=ax^3-2bx^2+(4+c)x+d-2,\quad g(x)=x^3-2x^2+ax-2a-3b+1,$$

where a, b, c and d are real parameters. Find the values of the parameters a, b, c and d, for which each of the functions has in the point $x=1$ a local minimum equal to 5.

Solution: It follows from the condition that:

$$\begin{cases} f(1)=a-2b+c+d+2=5 \\ g(1)=-a-3b=5 \\ f'(1)=3a-4b+c+4=0 \\ g'(1)=-1+a=0 \end{cases}$$

The solution of the system is: $a=1; b=-2; c=-15; d=13$.

Indeed, each of the functions $f(x)=x^3+4x^2-11x+11$ and $g(x)=x^3-2x^2+x+5$ has a local minimum in the point $x=1$, because $f'(1)=0$ and $f''(1)=12.1+8=20>0$, $g'(1)=0$ and $g''(1)=12.1-4=8>0$.

Problem 3. The function $f(x)=x^{2021}-x^{2020}+1$ and the matrix $A=\begin{pmatrix} -1 & -2 & 2 \\ 4 & 5 & -4 \\ 3 & 3 & -2 \end{pmatrix}$ are given. Find $f(A)$.

Solution: $A^2=\begin{pmatrix} -1 & -2 & 2 \\ 4 & 5 & -4 \\ 3 & 3 & -2 \end{pmatrix}=A$, so $A^n=A$, $n\in\mathbb{N}$. Therefore $f(A)=A^{2021}-A^{2020}+E$ i.e. $f(A)=A-A+E=E$ (E is the identity matrix of a third order).

Problem 4. The function $f(x)=\dfrac{\ln x}{x}$ is given.

a) Find the intervals in which $f(x)$ is monotonic.

b) Prove that $f(x)\le\dfrac{1}{e}$ for each $x>0$.

c) Determine which of the numbers 2020^{2021} or 2021^{2020} is greater.

Solution: a) The function is defined for $x\in(0,+\infty)$ and its derivative $f'(x)=\dfrac{1-\ln x}{x^2}=0$ when $x=e$. In the interval $(0,e)$ $f'(x)>0$ and $f(x)$ is increasing, in the interval $(e,+\infty)$ $f'(x)<0$ and $f(x)$ is decreasing.

b) From a) we obtain $\underset{x>0}{MAX}\ f(x)=f(e)=\dfrac{1}{e}$. Then $f(x)\le\underset{x>0}{MAX}\ f(x)=\dfrac{1}{e}$ for each

$x > 0$.

c) The function $f(x)$ is decreasing in the interval of $(e, +\infty)$. Therefore

$$f(2020) = \frac{\ln 2020}{2020} > \frac{\ln 2021}{2021} = f(2021), \quad 2021.\ln 2020 > 2020.\ln 2021,$$

$$\ln 2020^{2021} > \ln 2021^{2020}, \quad 2020^{2021} > 2021^{2020}.$$