BULGARIAN NATIONAL MATHEMATICAL OLYMPIAD FOR UNIVERSITY STUDENTS

VARNA, 14-16TH MAY, 2021

GROUP B

Problem 1. The matrix $A=\begin{pmatrix}1 & 0 & 0\\ a & 1 & 0\\ a^k & -a & 1\end{pmatrix}$ is given, $a>0$ and $k>0$. Let n be a natural number.

a) Find A^n;

b) Find the values of n for which the sum of the elements of A^n has the largest value if $a=2$ and $k=2021$?

Solution: a) Sequentially we calculate

$$A^2=\begin{pmatrix}1 & 0 & 0\\ 2a & 1 & 0\\ 2a^k-a^2 & -2a & 1\end{pmatrix},\ A^3=\begin{pmatrix}1 & 0 & 0\\ 3a & 1 & 0\\ 3a^k-3a^2 & -3a & 1\end{pmatrix},\ A^4=\begin{pmatrix}1 & 0 & 0\\ 4a & 1 & 0\\ 4a^k-6a^2 & -4a & 1\end{pmatrix},$$

$$A^5=\begin{pmatrix}1 & 0 & 0\\ 5a & 1 & 0\\ 5a^k-10a^2 & -5a & 1\end{pmatrix}.$$

We can conclude that $A^n=\begin{pmatrix}1 & 0 & 0\\ na & 1 & 0\\ na^k-\dfrac{n(n-1)}{2}a^2 & -na & 1\end{pmatrix}$.

This result can be proved by induction.

b) When $a=2$ and $k=2021$ $f(n)=-2n^2+2\left(1+2^{2020}\right).n+3$.

The maximum of this function is obtained for $n_0=\dfrac{-2\left(1+2^{2020}\right)}{2.(-2)}=\dfrac{1+2^{2020}}{2}=2^{2019}+\dfrac{1}{2}$.

As the value of n is not an integer, the largest value of $f(n)$ is obtained when $n_1=n_0-\dfrac{1}{2}=2^{2019}$ and $n_1=n_0+\dfrac{1}{2}=2^{2019}+1$.

Notice: The problem also makes sense when $n\le 0$.

Problem 2. In the plane Oxy the line $g: 4x-5y+3=0$ and the points $A(2,-6)$ and $B(5,-2)$ are given. Find:

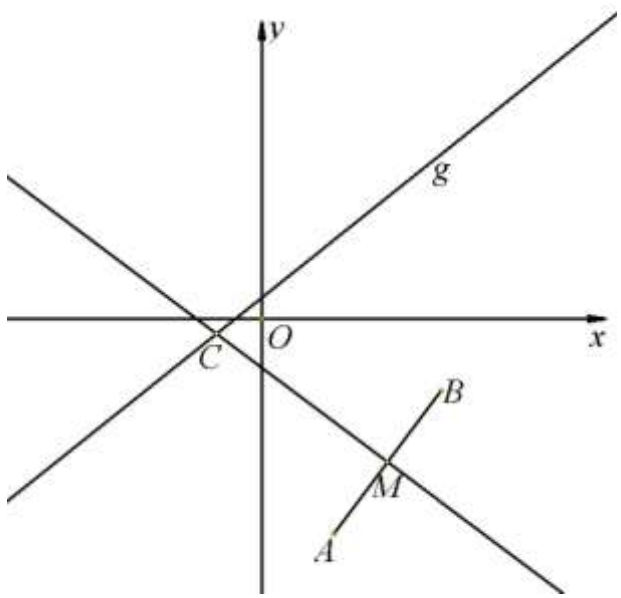
a) point C on the line g such that ΔABC is an isosceles triangle ($AC=BC$);

b) point D on the line g such that ΔABD has an area of 10;

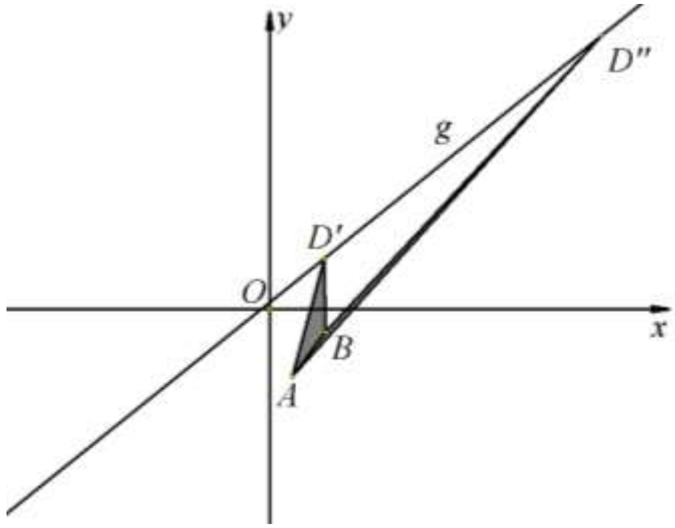
c) point P on the line g such that ΔABP has the smallest possible perimeter.

Solution: a) The point C lies on the bisector s of the line segment AB. The line s goes through the midpoint $M\left(\frac{7}{2},-4\right)$ of the line segment AB and it is perpendicular to the vector $\overrightarrow{AB}(3,4)$, so it has the following equation $s: 6x+8y+11=0$.

The coordinates of C are the solution of the system $\left|\begin{array}{l} g: 4x-5y+3=0 \\ s: 6x+8y+11=0 \end{array}\right. \Rightarrow C\left(-\frac{79}{62},-\frac{13}{31}\right)$.



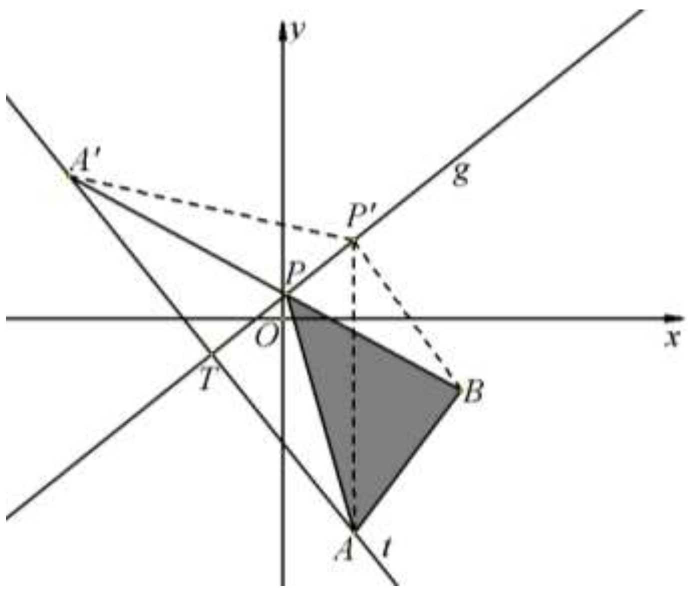
b) Let $D(x_D, y_D)$. The area of the triangle ΔABC $S_{ABD}=\frac{1}{2}\left|\begin{vmatrix} 2 & -6 & 1 \\ 5 & -1 & 1 \\ x_D & y_D & 1 \end{vmatrix}\right|=10$, which is equivalent to $-4x_D+3y_D+26\pm 20=0$. Since $D\in g$ the equality $4x_D-5y_D+3=0$ is satisfied. The system of the last two equations has two solutions: $D'\left(\frac{39}{8},\frac{9}{2}\right)$ and $D''\left(\frac{239}{8},\frac{49}{2}\right)$.



c) Let A' is the point, which is symmetric to *A* across *g*. Let *P* is the intersection of $A'B$ and *g*, and P' is an arbitrary point from *g*. If the perimeters of ΔABP and $\Delta ABP'$ are respectively *p* and p' then

$$p' = AB + AP' + BP' = AB + A'P' + BP' \geq AB + A'B = AB + A'P + PB = AB + AP + PB = p.$$

The equality $p' = p$ is satisfied then and only then $P' \equiv P$. Hence the sought point is $P = g \cap A'B$.



Let the line *t* goes through point $A(2,-6)$ and it is perpendicular to the line *g* $(t \perp \vec{n}(5,4))$. The equation of t is $5x+4y+14=0$. Therefore $\begin{vmatrix} g: 4x-5y+3=0 \\ t: 5x+4y+14=0 \end{vmatrix}$ and $T(-2,-1)$. From the coordinates of A and T we obtain $A'(-6,4)$ and the equation of the line $A'B$ is $A'B: 6x+11y-8=0$.

From the system $\left|\begin{array}{l} g: 4x-5y+3=0 \\ A'B: 6x+11y-8=0 \end{array}\right.$ we get $P\left(\frac{7}{74},\frac{25}{37}\right)$.

Problem 3. Prove the inequalities:

a) $\operatorname{tg} x \geq x$ if $x \in \left[0, \frac{\pi}{2}\right)$;

b) $\ln\left(\frac{2\cos x}{\sqrt{3}}\right) \leq \frac{\pi^2}{72} - \frac{x^2}{2}$ if $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$.

Solution: a) Let $f(x) = tgx - x$, $x \in \left[0, \frac{\pi}{2}\right)$.

Its derivative $f'(x) = \frac{1}{\cos^2 x} - 1 = tg^2 x > 0$. Therefore $f(x)$ is increasing and $f(x) \geq f(0) = 0$ i.e. $tgx \geq x$.

b) Let $f_1(u) = tgu$ and $f_2(u) = u$. From a) we have $f_1(u) \geq f_2(u)$, $\frac{\pi}{6} \leq u \leq x < \frac{\pi}{2} \Rightarrow$

$\int_{\frac{\pi}{6}}^{x} f_1(u)\,du \geq \int_{\frac{\pi}{6}}^{x} f_2(u)\,du$, $\int_{\frac{\pi}{6}}^{x} tgudu \geq \int_{\frac{\pi}{6}}^{x} udu$, $\int_{\frac{\pi}{6}}^{x} \frac{\sin u}{\cos u}\,du \geq \left.\frac{u^2}{2}\right|_{\frac{\pi}{6}}^{x}$, $-\int_{\frac{\pi}{6}}^{x} \frac{d\cos u}{\cos u} \geq \frac{x^2}{2} - \frac{\pi^2}{72}$,

$\left.\ln(\cos u)\right|_{\frac{\pi}{6}}^{x} \leq \frac{x^2}{2} - \frac{\pi^2}{72}$, $\ln(\cos x) - \ln\left(\cos\frac{\pi}{6}\right) \leq \frac{x^2}{2} - \frac{\pi^2}{72}$, $\ln(\cos x) - \ln\frac{\sqrt{3}}{2} \leq \frac{x^2}{2} - \frac{\pi^2}{72}$,

$\ln\left(\frac{2\cos x}{\sqrt{3}}\right) \leq \frac{x^2}{2} - \frac{\pi^2}{72}$.

Problem 4. Prove: $\lim\limits_{x\to\infty} \frac{1}{\sqrt{x^2+1}} \int_1^{x^2+1} \ln\left(1+\frac{1}{\sqrt{t}}\right) dt = 2$.

Solution: First we will calculate the integral $I = \int_1^{x^2+1} \ln\left(1+\frac{1}{\sqrt{t}}\right) dt$.

$$I = \int_1^{x^2+1} \ln\left(1+\frac{1}{\sqrt{t}}\right) dt = t.\ln\left(1+\frac{1}{\sqrt{t}}\right)\Bigg|_1^{x^2+1} - \int_1^{x^2+1} td\ln\left(1+\frac{1}{\sqrt{t}}\right) =$$

$$= \left(x^2+1\right)\ln\left(1+\frac{1}{\sqrt{x^2+1}}\right) - \ln 2 - \int_1^{x^2+1} \frac{t\left(-\frac{1}{2}\right)t^{-\frac{3}{2}}}{1+\frac{1}{\sqrt{t}}} dt =$$

$$=\left(x^2+1\right)\ln\left(1+\frac{1}{\sqrt{x^2+1}}\right)-\ln 2+\frac{1}{2}\int_1^{x^2+1}\frac{1}{1+\sqrt{t}}\,dt.$$

In the last integral we make a substitution $y=\sqrt{t}$ and we get

$$\int_1^{x^2+1}\frac{1}{1+\sqrt{t}}\,dt=\int_1^{\sqrt{x^2+1}}\frac{2y}{1+y}\,dy=2\int_1^{\sqrt{x^2+1}}\left(1-\frac{1}{1+y}\right)dy=2\left(\int_1^{\sqrt{x^2+1}}dy-\int_1^{\sqrt{x^2+1}}\frac{d\left(1+y\right)}{1+y}\right)=$$

$$=2\left(y\Big|_1^{\sqrt{x^2+1}}-\ln\left(1+y\right)\Big|_1^{\sqrt{x^2+1}}\right)=2\left(\sqrt{x^2+1}-1-\ln\sqrt{x^2+1}+\ln 2\right).$$

Therefore

$$I=\left(x^2+1\right)\ln\left(1+\frac{1}{\sqrt{x^2+1}}\right)-\ln 2-\frac{1}{2}.2.\left(\sqrt{x^2+1}-1-\ln\sqrt{x^2+1}+\ln 2\right)=$$

$$=\left(x^2+1\right)\ln\left(1+\frac{1}{\sqrt{x^2+1}}\right)-\ln 2-\sqrt{x^2+1}+1+\ln\sqrt{x^2+1}-\ln 2=$$

$$=\left(x^2+1\right)\ln\left(1+\frac{1}{\sqrt{x^2+1}}\right)-\sqrt{x^2+1}+1+\ln\sqrt{x^2+1}.$$

Now

$$\frac{1}{\sqrt{x^2+1}}\int_1^{x^2+1}\ln\left(1+\frac{1}{\sqrt{t}}\right)dt=\frac{1}{\sqrt{x^2+1}}I=\frac{x^2+1}{\sqrt{x^2+1}}\ln\left(1+\frac{1}{\sqrt{x^2+1}}\right)-\frac{1}{\sqrt{x^2+1}}+1+\frac{\ln\sqrt{x^2+1}}{\sqrt{x^2+1}}=$$

$$=\sqrt{x^2+1}\ln\left(1+\frac{1}{\sqrt{x^2+1}}\right)-\frac{1}{\sqrt{x^2+1}}+1+\frac{\ln\sqrt{x^2+1}}{\sqrt{x^2+1}}=$$

$$=\ln\left(1+\frac{1}{\sqrt{x^2+1}}\right)^{\sqrt{x^2+1}}-\frac{1}{\sqrt{x^2+1}}+1+\frac{\ln\sqrt{x^2+1}}{\sqrt{x^2+1}}.$$

Let $u=\sqrt{x^2+1}$. Then $u\to\infty$ and

$$L=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}}\int_1^{x^2+1}\ln\left(1+\frac{1}{\sqrt{t}}\right)dt=\lim_{u\to\infty}\left[\ln\left(1+\frac{1}{u}\right)^u-\frac{1}{u}+1+\frac{\ln u}{u}\right].$$

$\lim\limits_{u\to\infty}\ln\left(1+\frac{1}{u}\right)^u=\ln\lim\limits_{u\to\infty}\left(1+\frac{1}{u}\right)^u=\ln e=1$, $\lim\limits_{u\to\infty}\frac{1}{u}=0$ and $\lim\limits_{u\to\infty}\frac{\ln u}{u}=\lim\limits_{u\to\infty}\frac{\frac{1}{u}}{1}=\lim\limits_{u\to\infty}\frac{1}{u}=0$ hence $L=2$.